Reference Answer of Lecture 3

1. $\|x\|_* = \max_{t\in I}\left\|x(t)e^{-\lambda(t-t_0)}\right\|$, $\forall x\in C(I),\ \lambda > L,\ I=[t_0-h,t_0+h]$,Show the Picard theorem by Banach fixed point theorem.

Proof:

Only show the contraction condition.

For any $x_1, x_2 \in D$

$$\begin{aligned}
\|\mathrm{T}x_1-\mathrm{T}x_2\|_* &= \max_{t\in I}\left\|\mathrm{T}x_1e^{-\lambda(t-t_0)}-\mathrm{T}x_2e^{-\lambda(t-t_0)}\right\| \\
&= \max_{t\in I}\|\mathrm{T}x_1-\mathrm{T}x_2\|e^{-\lambda(t-t_0)} \\
&\le \max_{t\in I}\int_{t_0}^{t}\|f(s,x_1(s))-f(s,x_2(s))\|dse^{-\lambda(t-t_0)} \\
&\le L\max_{t\in I}\int_{t_0}^{t}\left\|[x_1(s)-x_2(s)]e^{-\lambda(s-t_0)}e^{\lambda(s-t_0)}\right\|dse^{-\lambda(t-t_0)} \\
&\le L\|x_1-x_2\|_*\max_{t\in I}\int_{t_0}^{t}e^{\lambda(s-t)}ds \\
&= L\|x_1-x_2\|_*\frac{1-e^{\lambda(t_0-t)}}{\lambda} \\
&\le \frac{L}{\lambda}\|x_1-x_2\|_* \qquad \frac{L}{\lambda}<1
\end{aligned}$$

Problem 12:

(a) Consider the norm of C[(0,a)] given by $\|f\|_e = \max_{0\le t\le a}|f(t)|e^{-t^2}$ .Let $\mathrm{T}f(t)=\int_0^t sf(s)ds$ .Show that $\|\mathrm{T}f\|_\infty \le \frac{a^2}{2}\|f\|_\infty$ and $\|\mathrm{T}f\|_e \le \frac{1}{2}\|f\|_e$.

(b) Show that the integral equation $x(t)=\frac{1}{2}t^2+\int_0^t sx(s)ds \quad t\in[0,a]$ has exactly one solution. Determine the solution (i) by rewriting the equation as an initial value problem and solving it, (ii) by using the methods of successive approximations starting with $x_0\equiv 0$.

Proof:

(a)

$$\|\mathrm{T}f\|_\infty = \max_{0\le t\le a}\left|\int_0^t sf(s)ds\right| \le \max_{0\le t\le a}\int_0^t|sf(s)|ds \le \|f\|_\infty\max_{0\le t\le a}\int_0^t|s|ds = \|f\|_\infty\max_{0\le t\le a}\frac{t^2}{2}\le\frac{a^2}{2}\|f\|_\infty$$

$$\|\mathrm{T}f\|_e = \max_{0\le t\le a}\left|\int_0^t sf(s)ds\right|e^{-t^2} \le \max_{0\le t\le a}\int_0^t\left|sf(s)e^{-s^2}e^{s^2}\right|dse^{-t^2} \le \|f\|_e\max_{0\le t\le a}\int_0^t\left|se^{s^2}\right|dse^{-t^2}$$

$$= \|f\|_e\max_{0\le t\le a}\frac{e^{t^2}-1}{2}e^{-t^2} = \|f\|_e\max_{0\le t\le a}\frac{1-e^{-t^2}}{2}\le\frac{1}{2}\|f\|_e$$

(b) $x'(t)=t+tx(t)$ $\therefore f(x,t)=t+tx$ Obviously it's continuous.

Then $\|f(t,x_1(t))-f(t,x_2(t))\|=\|t(x_1-x_2)\|\le a\|x_1-x_2\|$

$\therefore f(x,t)$ satisfies the Liptischz condition,so $f(x,t)$ satisfies Picard theorem.

Then the equation has exactly one solution.

(i) To solve $x'(t)=t+tx(t)$,we have proved in Lecture 2 that the general solution of ODE

$$x'(t)=\mathrm{Q}(t)+\mathrm{P}(t)x(t) \text{ is } x(t)=(C(t_0)+\int_{t_0}^{t}\mathrm{Q}(s)e^{-\int_{t_0}^{s}P(\tau)d\tau}ds)e^{\int_{t_0}^{t}P(s)ds}$$

Here, $\mathrm{P(t)=t,Q(t)=t,t_0}=0$

For $C(0)=x(0)=0$,then $x(t)=(\int_0^t se^{-\int_0^s\tau d\tau}ds)e^{\int_0^t sds}=e^{\frac{t^2}{2}}-1$.

$$x_0(t)=0$$

$$x_n(t)=x_0(t)+\int_0^t f(s,x_{n-1}(s))ds=\int_0^t(s+sx_{n-1}(s))ds$$

$$x_1(t)=\int_0^t(s+s\cdot 0)ds=\frac{t^2}{2}$$

(ii) $x_2(t)=\int_0^t(s+s\cdot\frac{s^2}{2})ds=\frac{t^2}{2}+\frac{t^4}{8}$

$$x_3(t)=\int_0^t(s+s\cdot(\frac{s^2}{2}+\frac{s^4}{8}))ds=\frac{t^2}{2}+\frac{t^4}{8}+\frac{t^6}{48}$$

$\vdots$

$x_n(t)=\sum_{i=1}^{n}\frac{t^{2i}}{2^i i!}$ is the Taylor expension of $e^{\frac{t^2}{2}}-1$

So $x(t)=\lim_{n\to\infty}x_n(t)=e^{\frac{t^2}{2}}-1$